Reconciling The Change In Portfolio Value Over Time

Gary Schurman, MBE, CFA

December, 2020

In this white paper we will build a model to reconcile the dollar change in portfolio value over time. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to reconcile the change in portfolio value over time. We are given the following go-forward model assumptions...

Table 1: Fund Go-Forward Assumptions

Symbol	Description	Value
P_0	Fund value at time zero ($)	10,000,000
μ	Expected annual return (%)	10.00
σ	Expected return volatility (%)	20.00
λ	Annualized fees and expenses (%)	2.00
ϕ	New investment (i.e. reinvestment) (%)	6.00
ϵ	Sales of investments (i.e. distributions) (%)	4.00
T	Term in years (#)	3.00

Our task is to answer the following questions:

Question 1: What is portfolio value at the end of year three given that the random variable drawn from a normal distribution is 0.75?

Question 2: Reconcile the change in portfolio value over the three year period.

Note: Assume that the rate variables in the table above are continuous-time rates.

Modeling Portfolio Value Over Time

We will define the variable δW_t to be the change in the underlying brownian motion over time. Using the parameters in Table 1 above the stochastic differential equation for the change in portfolio value over the time interval $[t, t+\delta t]$ is...

$$\delta P_t = \mu\, P_t\, \delta t - \lambda\, P_t\, \delta t + \phi\, P_t\, \delta t - \epsilon\, P_t\, \delta t + \sigma\, P_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

Note that we can normalize and rewrite Equation (1) above as...

$$\delta P_t = \mu\, P_t\, \delta t - \lambda\, P_t\, \delta t + \phi\, P_t\, \delta t - \epsilon\, P_t\, \delta t + \sigma\, P_t \sqrt{t}\, Z \text{ ...where... } Z \sim N\Big[0, 1\Big] \tag{2}$$

The solution to Equation (2) above is the equation for portfolio value at the end of time T, which is...

$$P_T = P_0 \operatorname{Exp}\left\{\left(\mu - \lambda + \phi - \epsilon - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\, Z\right\} \text{ ...where... } Z \sim N\Big[0, 1\Big] \tag{3}$$

We will define the variable θ to be the normally-distributed random net annualized rate of return. Using Equation (3) above the equation for the variable θ is...

$$\theta = \left[\left(\mu - \lambda - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\, Z\right] \Big/ T \text{ ...where... } Z \sim N\Big[0, 1\Big] \tag{4}$$

Using Equation (4) above we can rewrite portfolio value Equation (3) above as...

$$P_T = P_0 \, \text{Exp}\left\{ \left(\theta + \phi - \epsilon \right) T \right\} \text{ ...where... } \theta \, T \sim N\left[\left(\mu - \lambda - \frac{1}{2} \, \sigma^2 \right) T, \sigma^2 T \right] \tag{5}$$

Using Equation (4) above we can rewrite the Equation (2) above as...

$$\delta P_t = \theta \, P_t \, \delta t + \phi \, P_t \, \delta t - \epsilon \, P_t \, \delta t \text{ ...where... } \theta \, \delta t \sim N\left[\left(\mu - \lambda - \frac{1}{2} \, \sigma^2 \right) \delta t, \sigma^2 \delta t \right] \tag{6}$$

Reconciling the Change in Portfolio Value

Note that we can rewrite portfolio value Equation (15) above as...

$$P_T = P_0 + \int_0^T \delta P_u \, \delta u \tag{7}$$

Using Equation (6) above we can rewrite Equation (7) above as...

$$P_T - P_0 = \int_0^T \theta \, P_u \, \delta u + \int_0^T \phi \, P_u \, \delta u - \int_0^T \epsilon \, P_u \, \delta u \tag{8}$$

Using Equation (15) above we can rewrite the first integral in Equation (8) above as...

$$\int_0^T \theta \, P_u \, \delta u = \theta \int_0^T P_0 \, \text{Exp}\left\{ \left(\theta + \phi - \epsilon \right) u \right\} \delta u \tag{9}$$

Using Equation (15) above the solution to the integral in Equation (9) above is...

$$\int_0^T P_0 \, \text{Exp}\left\{ \left(\theta + \phi - \epsilon \right) u \right\} \delta u = \frac{1}{\theta + \phi - \epsilon} \, P_0 \left[\text{Exp}\left\{ \left(\theta + \phi - \epsilon \right) T \right\} - 1 \right] = \frac{1}{\theta + \phi - \epsilon} \left(P_T - P_0 \right) \tag{10}$$

Using Equations (8) and (10) above the equation for portfolio dollar return or loss over the time interval $[0, T]$ is...

$$\text{Portfolio dollar return or loss} = \int_0^T \theta \, P_u \, \delta u = \frac{\theta}{\theta + \phi - \epsilon} \left(P_T - P_0 \right) \tag{11}$$

Using Equations (8) and (10) above the equation for portfolio dollar reinvestment over the time interval $[0, T]$ is...

$$\text{Portfolio dollar reinvestment} = \int_0^T \phi \, P_u \, \delta u = \frac{\phi}{\theta + \phi - \epsilon} \left(P_T - P_0 \right) \tag{12}$$

Using Equations (8) and (10) above the equation for portfolio dollar sales over the time interval $[0, T]$ is...

$$\text{Portfolio dollar sales} = \int_0^T \epsilon \, P_u \, \delta u = \frac{\epsilon}{\theta + \phi - \epsilon} \left(P_T - P_0 \right) \tag{13}$$

The Answers To Our Hypothetical Problem

Question 1: What is portfolio value at the end of year three given that the random variable drawn from a normal distribution is 0.75?

Using Equation (4) the equation for the random variable theta is...

$$\theta = \left[\left(0.10 - 0.02 - \frac{1}{2} \times 0.20^2\right) \times 3.00 + 0.20 \times \sqrt{3.00} \times 0.75\right] \Big/ 3.00 = 0.14660 \tag{14}$$

Using Equations (15) and (14) above the answer to the question is...

$$P_3 = 10,000,000 \times \text{Exp}\left\{\left(0.14660 + 0.06 - 0.04\right) \times 3\right\} = 16,484,041 \tag{15}$$

Question 2: Reconcile the change in portfolio value over the three year period.

Using Equation (11) above the equation for portfolio dollar return or loss over the time interval $[0, 3]$ is...

$$\text{Portfolio dollar return or loss} = \frac{0.14660}{0.14660 + 0.06 - 0.04}\left(16,484,041 - 10,000,000\right) = 5,705,657 \tag{16}$$

Using Equation (12) above the equation for portfolio dollar reinvestment over the time interval $[0, 3]$ is...

$$\text{Portfolio dollar reinvestment} = \frac{0.06}{0.14660 + 0.06 - 0.04}\left(16,484,041 - 10,000,000\right) = 2,335,153 \tag{17}$$

Using Equation (13) above the equation for portfolio dollar sales over the time interval $[0, 3]$ is...

$$\text{Portfolio dollar sales} = \frac{-0.04}{0.14660 + 0.06 - 0.04}\left(16,484,041 - 10,000,000\right) = -1,556,769 \tag{18}$$